PMM U.S.S.R.,Vol.51,No.2,pp.156-161,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00


# THE PROBLEM OF THE NASH EQUILIBRIUM SITUATION IN A POSITIONAL n-PERSON GAME WITH A MEMORY*

YU.E. CHISTYAKOV

The approach of a pencil of trajectories to the terminal set, when the control side uses positional strategies with a memory and is ignorant of the discrete parameter $i \in I$, on which both the dynamic system and the terminal set depend, is considered. The set of initial positions for which the problem is solvable is constructed. It is shown that this set can be constructed by solving several standard problems on approach in positional strategies without a memory and without undetermined parameters. It is also shown that the set has in a certain sense the bridge property which is typical for the solution of positional approach problems with undertermined parameters /1, 2/. By solving our problem, the results of /3/ on the necessary and sufficient conditions for the existence of an equilibrium situation in positional two-person differential games can be extended to the $n$-person case. The role of undertermined parameter is then played by the number of the player who deviates from equilibrium.

**1. The approach problem when an undetermined parameter is present.** The approach problem, when a discrete parameter is unknown, is specified by the system

$$dx/dt \in F(i, x(t), t, u'(t)), \; x(t_0) = x_0$$
$$u'(t) \equiv u(x(t_0, \tau_k)), \; t \in [\tau_k, \tau_{k+1}), \; \tau_k \in \Delta \tag{1.1}$$

We wish to ensure that $x(\vartheta) \in M(i)$, $\forall\, i \in I$. Here, $x \in X = R^l$, $t_0 \in T = [\vartheta_0, \vartheta]$, $I = \{1, 2, \ldots, n\}$, $M(i) \subset X$ $u'(t) \in P$, $\Delta = \{\tau_k, k = 0, 1, \ldots, p' \mid t_0 = \tau_0 < \tau_1 < \ldots < \tau_{p'} = \vartheta\}$.

In other words, the phase space is finite-dimensional Euclidean space $X$ and the game is played in the time interval $[t_0, \vartheta]$. The control side (player) uses piecewise constant positional strategies with a memory, which are specified by the pair $(u, \Delta)$. Here, $u : X[t_0, \cdot] \to P$ is a function which associates with each trajectory $x(t_0, t)$, realized at instant $t$, a value of the control $u'(t) = u(x(t_0, t))$ of compactum $P$ in topological space; $\Delta$ are the instants at which the control side "switches" its control; $M(i)$ are bounded sets which depend on the value of the unknown parameter $i \in I$. For each fixed set $i, x, t, u'$ the many-valued function $F$ takes the value $F(i, x, t, u')$, which is a bounded closed set in the sphere, radius $R$, of $X$. We assume that, for all subsets $\sigma \subseteq I$, the many-valued function

$$F(\sigma, x, t, u') = \bigcap_{i \in \sigma} F(i, x, t, u')$$

is continuous with respect to $t, u'$ and satisfies a Lipschitz condition with respect to $x$ in the Hausdorff metric. Notice that, if $F(\sigma, x, t, u') = \varnothing$ for certain $\sigma, x, t, u'$, it must be empty for all $x', t', u''$.

If we are given the initial position $x_0, t_0$, the value of the unknown parameter $i \in I$, and the strategy $u\Delta$, then system (1.1) uniquely generates a pencil of absolutely continuous trajectories $X[i, x_0, t_0, u\Delta]$, which satisfy the first equation in (1.1) for almost all $t \in [t_0, \vartheta]$.

*Definition 1.1.* We shall say that the approach problem is solvable for $\sigma \subseteq I$ and initial position $x_0, t_0$ if, given any $\delta > 0$, there is a strategy $u\Delta$ which ensures that the entire pencil $X[i, x_0, t_0, u\Delta]$ hits at instant $\vartheta$ the $\delta$-neighbourhood of the set $M(i)$, where the inclusion must hold for all $i \in \sigma$.

Denote by $K(\sigma) \subset X \times T$ the set of initial positions $x_0, t_0$, for which the approach problem is solvable with fixed $\sigma \subseteq I$, or formally,

$$K(\sigma) = \{x_0, t_0 \mid \forall \delta > 0 \;\; \exists u\Delta\colon X^\vartheta[i, x_0, t_0, u\Delta] \subset M(i) + \delta, \;\; \forall i \in \sigma\} \tag{1.2}$$

*Prikl.Matem.Mekhan.,51,2,201-208,1987

Here and below, $X^{\vartheta}[i, x_0, t_0, u\Delta]$ is the pencil of trajectories cut off at instant $\vartheta$, and $M(i) + \delta$ is the $\delta$-neighbourhood of the set $M(i)$.

Notice that, if $\sigma$ consists of a single element, our problem reduces to the well-known approach problem /1/. In particular, the corresponding set $K(\sigma)$ has the bridge property, of which essential use is made for constructing the strategies $u\Delta$ which ensure encounter with the terminal set.

We will study the properties of sets $K(\sigma)$ with $|\sigma| > 1$, give a method for constructing these sets, and find the strategies $u\Delta$ which solve our problem. Our main interest is in the set $K(I)$ and the corresponding strategies.

*Properties of the sets* $K(\sigma)$. We shall show that the set $K(\sigma)$ has a number of typical properties of bridges in approach problems without unknown parameters.

We note without proof the two following topological properties of these sets.

*Property 1.1.* The set $K(\sigma)$ is a closed subset of the extended phase space $X \times T$.

*Property 1.2.* There is a quantity $\varepsilon_0(\delta) \to 0$ as $\delta \to 0$ such that $K(\sigma, \delta) \subset K(\sigma) + \varepsilon_0(\delta)$, where the set $K(\sigma, \delta)$ is given by (1.2) for fixed $\delta > 0$.

Property 1.2 shows that, for small $\delta$ the set $K(\sigma, \delta)$ "deviates little" from $K(\sigma)$.

If we are given $\sigma \subseteq I$, the initial position $x_0, t_0$, and strategy $u\Delta$, we can consider, instead of pencils of trajectories $X[i, x_0 t_0, u\Delta]$ the pencil

$$X[\sigma, x_0, t_0, u\Delta] = \bigcap_{i\in\sigma} X[i, x_0, t_0, u\Delta] \tag{1.3}$$

This is the set of trajectories which could be realized for any value of the parameter $i \in \sigma$.

It follows at once from the definition of system (1.1) that this set is the same as the set of absolutely continuous solutions of the differential equation in contingencies

$$dx/dt \in F(\sigma, x, t, u(x(t_0, \tau_k))), \quad t \in [\tau_k, \tau_{k+1}), \quad x(t_0) = x_0 \tag{1.4}$$

where, if the set $F(\sigma, x, t, u)$ happens to be identically equal to the empty set for some $\sigma$ and all $x, t, u'$, then the solution of Eq.(1.4) and the corresponding pencil (1.3) consist by definition of the single trajectory $x(t_0, t_0)$, which is simply the initial position $(x_0, t_0)$.

It turns out that set $K(\sigma)$ is a bridge for these pencils.

*Property 1.3.* If $(x_0, t_0) \in K(\sigma)$, then, given any $\varepsilon > 0$, there is a strategy $u\Delta$ such that $\bar{X}[\sigma, x_0, t_0, u\Delta] \subset K(\sigma) + \varepsilon$. (Here and below, the bar above a trajectory shows that it has to be regarded as a set of points in space $X \times T$, through which it passes).

*Proof.* Given $\varepsilon > 0$, let $u\Delta$ be a solution of problem (1.2) for fixed $\delta > 0$, chosen in such a way that $\varepsilon_0(\delta) < \varepsilon/2$ (see Property 1.2). We shall assume that $R\|\Delta\| < \varepsilon/2$, where $\|\Delta\| = \max_k (\tau_{k+1} - \tau_k)$. Let $x_0(t_0, \vartheta) \in X[\sigma, x_0, t_0, u\Delta]$ and $x_p = x_0^{\tau_p}(t_0, \vartheta)$ is the value of the phase vector of trajectory $x_0(t_0, \vartheta)$ at the instant $\tau_p \in \Delta$. We shall show that then $(x_p, \tau_p) \in K(\sigma) + \varepsilon/2$.

For all trajectories starting at $(x_p, \tau_p)$ we define $u_1(x(\tau_p, t)) = u(\varphi(x(\tau_p, t)))$, where $\varphi(x(\tau_p, t)) = x_0(t_0, \tau_p) * x(\tau_p, t)$.

For asterisk denotes the "sewing together" of the trajectories at the point $(x_p, \tau_p)$. It then follows from (1.1) that

$$x_0(t_0, \tau_p) * X[i, x_p, \tau_p, u_1\Delta] \subset X[i, x_0, t_0, u\Delta], \quad \forall i \in \sigma$$

Consequently, $X^{\vartheta}[i, x_p, \tau_p, u_1\Delta] \subset X^{\vartheta}[i, x_0, t_0, u\Delta] \subset M(i) + \delta$, since $u\Delta$ solves (1.2). Hence, by definition,

$$(x_p, \tau_p) \in K(\sigma, \delta) \subset K(\sigma) + \varepsilon_0(\delta) \subset K(\sigma) + \varepsilon/2$$

Now let $t \in (\tau_p, \tau_{p+1})$. We then have

$$(x^t(t_0, \vartheta), t) \in (x_p, \tau_p) + R\|\Delta\| \subset K(\sigma) + \varepsilon$$

which proves property 1.3.

*Construction of sets* $K(\sigma)$. We shall show that the sets $K(\sigma)$ can be obtained as solutions of the bridge construction problem

$$W(\sigma) = \begin{cases} x_0, t_0 \mid \forall\delta > 0\ \exists u\Delta:\ X^{\vartheta}[j, x_0, t_0, u\Delta] \subset M(j) + \delta, & \sigma = \{j\} \\ x_0, t_0 \mid \forall\varepsilon > 0\ \exists u\Delta:\ \bar{X}[\sigma, x_0, t_0, u\Delta] \subset \bigcap_{i\in\sigma} W(\sigma \setminus i) + \varepsilon, & |\sigma| > 1 \end{cases} \tag{1.5}$$

Thus, if $\sigma$ consists of a single element $j$, then, in order to construct $W(\sigma)$ we have to solve the approach problem with the set $M(j)$. If $\sigma$ consists of more than one element, then, to construct $W(\sigma)$, we have to solve the problem of keeping the pencil $X[\sigma, x_0, t_0, u\Delta]$, generated by system (1.4), inside the intersection of the previously constructed sets $W(\sigma')$, $\sigma' = \sigma \setminus i$,

$i \in \sigma$.

Notice that there are no undetermined parameters in these problems, so that information about the entire trajectory is redundant, and we can manage with the usual positional strategies without a memory to construct sets $W(\sigma)$. Thus the sets $W(\sigma)$, being the solutions of positional problems, will have the bridge property, as indicated by:

*Property 1.4.* For each fixed $\delta > 0$, there are positional strategies without a memory $v_\delta^\sigma \Delta_\delta$ $(\sigma \subseteq I)$ and a sequence $0 < \varepsilon_n(\delta) < \varepsilon_{n-1}(\delta) < \ldots < \varepsilon_1(\delta)$ $(n = |I|)$, such that $\forall \sigma \subseteq I, (x_0, t_0) \in W(\sigma) + \varepsilon_m(\delta)$, where $m = |\sigma|$, we have the condition

$$X^\vartheta[j, x_0, t_0, v_\delta^j \Delta_\delta] \subset M(j) + \delta, \quad \sigma = \{j\}$$
$$\bar{X}[\sigma, x_0, t_0, v_\delta^\sigma \Delta_\delta] + R\,\|\Delta_\delta\| \subset W(\sigma) + \varepsilon_{m-1}, \; |\sigma| = m > 1 \tag{1.6}$$

Now, using positional strategies $v_\delta^\sigma \Delta_\delta$, ensuring inclusion (1.6), we construct positional strategies with a memory $u_\delta^{\sigma_0} \Delta_\delta'$, which will solve for fixed $\delta > 0$ our original problem (1.2) for all $(x_0, t_0) \in K(\sigma_0)$.

In short, given $\delta > 0$, $\sigma_0 \subseteq I$. We find strategy $u_\delta^{\sigma_0} \Delta_\delta'$, by putting $\Delta_\delta' = \Delta_\delta$, $u_\delta^{\sigma_0}(x(t_0, t)) = v_\delta^{\sigma_p}(x^t, t)$, where $x^t = x^t(t_0, t)$, and we find the set $\sigma_p = \psi_\delta(x(t_0, t), \sigma_0) \subseteq \sigma_0$ as a result of the sequential procedure

$$\sigma_{k+1} = \{j \in \sigma_k \mid x(\tau_k, \tau_{k+1}) \in X[j, x_k, \tau_k, v_\delta^{\sigma_k} \Delta_\delta]\}, \quad k = 0, 1, \ldots, p-1$$
$$x_k = x^{\tau_k}(t_0, t), \quad \tau_k \in \Delta_\delta, \quad t \in [\tau_p, \tau_{p+1})$$
$$x(t_0, t) = x(t_0, \tau_1) * x(\tau_1, \tau_2) * \ldots * x(\tau_p, t)$$

By definition of strategy $u_\delta^{\sigma_0} \Delta_\delta$, it clearly associates with each trajectory $x(t_0, t)$ a value equal to the value taken by positional strategy $v_\delta^{\sigma_p} \Delta_\delta$ at the end point $(x^t(t_0, t), t)$ of this trajectory. We then compute $\sigma_p$ recurrently at the instant $t \in [\tau_p, \tau_{p+1})$ and indicate the complete set of unknown parameters of $\sigma_0$ for which this trajectory could be realized.

In particular, if $(x_0, t_0) \in W(\sigma_0) + \varepsilon_m$, $m = |\sigma_0|$, $i \in \sigma_0$, $x(t_0, \vartheta) \in X[i, x_0, t_0, u_\delta^{\sigma_0} \Delta_\delta]$, $\psi_\delta(x(t_0, t), \sigma_0) = \sigma_0$, then it follows at once from the definition that the strategy $v_\delta^{\sigma_0} \Delta_\delta$, for which inclusion (1.6) holds acts for the entire time on trajectory $x(t_0, t)$. Consequently,

$$x^\vartheta(t_0, \vartheta) \in M(j) + \delta, \quad \sigma_0 = \{j\}$$
$$\bar{x}(t_0, \vartheta) + R\,\|\Delta_\delta\| \subset W(\sigma_0) + \varepsilon_{m-1}, \; |\sigma_0| = m > 1 \tag{1.7}$$

*Theorem 1.1.* We have the equation $K(\sigma_0) = W(\sigma_0)$, where strategy $u_\delta^{\sigma_0} \Delta_\delta$ ensures the appropriate inclusion in (1.2) for fixed $\delta > 0$ and for all initial positions $(x_0, t_0) \in K(\sigma_0) + \varepsilon_m(\delta)$, where $m = |\sigma_0|$.

*Proof.* With $\sigma_0 = \{i\}$, sets $K(j)$ and $W(j)$ are equal by definition. Now let $(x_0, t_0) \in K(j) + \varepsilon_1(\delta)$. Condition (1.6) then holds, and hence, (1.2). Assume that the theorem is true for $|\sigma_0| < m$. We shall show that it then holds for $|\sigma_0| = m > 1$.

We first show that $K(\sigma_0) \subseteq W(\sigma_0)$. If $(x_0, t_0) \in K(\sigma_0)$, then, by property 1.3, given any $\varepsilon > 0$, there is a strategy $u\Delta$ such that

$$\bar{X}[\sigma_0, x_0, t_0, u\Delta] \subset K(\sigma_0) + \varepsilon \subseteq \bigcap_{i \in \sigma_0} K(\sigma_0 \setminus i) + \varepsilon = \bigcap_{i \in \sigma_0} W(\sigma_0 \setminus i) + \varepsilon$$

The last equation follows from the inductive assumption, so that, in accordance with the definition, $(x_0, t_0) \in W(\sigma_0)$.

We shall prove the reverse inclusion: $W(\sigma_0) \subseteq K(\sigma_0)$. It suffices to show that, if $(x_0, t_0) \in W(\sigma_0) + \varepsilon_m(\delta)$, $x(t_0, \vartheta) \in X[i, x_0, t_0, u_\delta^{\sigma_0} \Delta_\delta]$ for some $i \in \sigma_0$, then $x^\vartheta(t_0, \vartheta) \in M(i) + \delta$.

The first case is $\psi_\delta(x(t_0, \vartheta), \sigma_0) = \sigma$. Then, in accordance with (1.7).

$$\bar{x}(t_0, \vartheta) \subset W(\sigma_0) + \varepsilon_{m-1} \subseteq \bigcap_{j \in \sigma_0} W(\sigma_0 \setminus j) + \varepsilon_{m-1}$$

By the inductive assumption, the last set is the same as the set

$$\bigcap_{j \in \sigma_0} K(\sigma_0 \setminus j) + \varepsilon_{m-1} \subset K(i) + \varepsilon_1$$

Consequently, $x^\vartheta(t_0, \vartheta) \in M(i) + \delta$, since, by property 1.4, the section by the instant $\vartheta$ of set $K(i) + \varepsilon_1$ belongs to $M(i) + \delta$.

The second case is $\psi_\delta(x(t_0, \vartheta), \sigma_0) = \sigma' \subset \sigma_0$. Let $p = \max\{k \mid \psi_\delta(x(t_0, \tau_k), \sigma_0) = \sigma_0\}$. Then, $\sigma_p = \sigma_0$, $\sigma_{p+1} \subset \sigma_0$, $|\sigma_{p+1}| = m' < m$, $\varepsilon_{m'} \geqslant \varepsilon_{m-1}$. It follows from relations (1.7) that

$$\bar{x}(t_0, \tau_p) + R\,\|\Delta_\delta\| \subset \bigcap_{j \in \sigma_0} W(\sigma_0 \setminus j) + \varepsilon_{m-1}$$

Since $\sigma_{p+1} \subset \sigma_0$, we have $\sigma_{p+1} \subseteq \sigma_0 \setminus j$ for some $j$, so that the latter set belongs to $W(\sigma_{p+1}) + \varepsilon_m{}'$. Consequently,

$$(x_{p+1}, \tau_{p+1}) \in (x_p, \tau_p) + R \cdot \| \Delta_\delta \| \subset W(\sigma_{p+1}) + \varepsilon_{m'}.$$

Thus the condition of the theorem hold for position $(x_{p+1}, \tau_{p+1})$, but only for $\sigma_{p+1} \subset \sigma_0$. Moreover, since strategy $u_\delta^{\sigma_{p+1}} \Delta_\delta$ acts in the interval $(\tau_{p+1}, \vartheta)$, then (1.2) holds by the inductive assumption, i.e., $x^\vartheta(t_0, \vartheta) \in M(i) + \delta$, which proves the theorem.

**2. The problem of the equilibrium situation in a positional $n$-person game.** The game is specified by the system

$$\begin{gathered} dx/dt = f(x, t, u_1'(t), \ldots, u_n'(t)), \; x(t_0) = x_0 \\ u_i'(t) = u_i(x(t_0, \tau_k^i)), \quad t \in [\tau_k^i, \tau_{k+1}^i), \quad \tau_k^i \in \Delta_i \\ J_i(u_1\Delta_1, \ldots, u_n\Delta_n) = g_i(x(\vartheta)) \to \max \; i \in I = \{1, 2, \ldots, n\} \end{gathered} \tag{2.1}$$

The game is played in a finite-dimensional space $X$ in the time interval $[t_0, \vartheta] \subseteq [\vartheta_0, \vartheta] = T$. The players use piecewise-constant positional strategies with a memory, specified by the pair $(u_i, \Delta_i)$, $u_i : X[t_0, \cdot] \to P_i$ is a function which associates with each trajectory $x(t_0, t)$, realized at instant $t$, a value of control $u_i'(t)$ of the compactum $P_i$ in topological space $\Delta_i \rightleftharpoons \{\tau_k^i, k = 0, 1, \ldots, p_i \mid t_0 = \tau_0^i < \tau_1^i < \ldots < \tau_{p_i}^i = \vartheta\}$ are the instants at which the $i$-th player intends to "switch" his control. The function $f$, which defines the game dynamics, is assumed to be continuous with respect to its arguments, and satisfies the inequality $\varkappa(1 + \|x\|) \geqslant \|f(x, t, u_1', \ldots, u_n')\|$, for some $\varkappa > 0$, and a Lipschitz condition with respect to $x$. Moreover, the many-valued function

$$F(\sigma, x, t, u_1', \ldots, u_n') = \bigcap_{i \in \sigma} \{f(x, t, u_1', \ldots, u_i'', \ldots, u_n') \mid u_i'' \in P_i\} \tag{2.2}$$

constructed on the basis of the function $f$, is continuous with respect to $u' = (u_1', \ldots, u_n')$, $t$ and satisfies a Lipschitz condition with respect to $x$ in the Hausdorff metric for every $\sigma \subseteq I$.

If we are given the initial position $x_0$, $t_0$ and the set of player's strategies $u\Delta = (u_1\Delta_1, \ldots, u_n\Delta_n)$, then system (2.1) uniquely generates a trajectory $x(\cdot)$, and along with it, the pay-off's are given by the third expression in (2.1).

*Definition 2.1.* For initial position $(x_0, t_0)$ in game (2.1) we have an equilibrium situation if, given any $\varepsilon > 0$, there is a strategy $u\Delta = (u_1\Delta_1, \ldots, u_n\Delta_n)$ such that, for all $i \in I$, we have the inequalities

$$J_i(u_1\Delta_1, \ldots, u_i''\Delta_i'', \ldots, u_n\Delta_n) \leqslant J_i(u_1\Delta_1, \ldots, u_i\Delta_i, \ldots, u_n\Delta_n) + \varepsilon, \quad \forall u_i''\Delta_i'' \tag{2.3}$$

It can be shown that the existence of an equilibrium trajectory (ET) is uniquely associated with the existence of an equilibrium situation.

*Definition 2.2.* Trajectory $x^*(\cdot)$ is the equilibrium one for differential game (2.1) if, given any $\varepsilon > 0$, there is a set of strategies $u\Delta$, generating trajectory $x(\cdot)$, for which, first, we have the inequality

$$\max_{t_0 \leqslant t \leqslant \vartheta} \| x(t) - x^*(t) \| \leqslant \varepsilon \tag{2.4}$$

and second, we have inequality (2.3) of Definition 2.1.

Our subject of further study is the ET's which, being limit elements of $\varepsilon$-equilibrium trajectories, have several good mathematical properties.

*The necessary and sufficient condition for the trajectory to be an equilibrium one.* It will be shown below that the problem of ET construction amounts to solving an auxiliary approach problem in the presence of an undetermined parameter, as described in Sect.1.

We shall first define the ET in a more convenient form. It can be shown that the set of ET's remains unchanged if we require in Definition 2.1 that the divisions $\Delta_i$ of all players be the same, while the deviating player chooses all possible measurable programs. On the basis of this fact, the theorems on measurable choice, and on the continuity of the terminal pay-off functions, can provide the following equivalent definition of ET.

*Definition 2.3.* Trajectory $x^*(\cdot)$ is an equilibrium trajectory if, given any $\varepsilon > 0$, there is a set of strategies $u\Delta = (u_1\Delta, \ldots, u_n\Delta)$, generating trajectory $x(\cdot)$, for which, first, condition (2.4) holds, and second, for each $i \in I$, the entire pencil $X[i, x_0, t_0, u\Delta]$ hits at instant $\vartheta$ the $\varepsilon$-neighbourhood of the corresponding set

$$M(i) = \{x \mid g_i(x) \leqslant g_i(x^*(\vartheta))\} \tag{2.5}$$

The definition of the pencil $X[i, x_0, t_0, u\Delta]$ was given in Sect.1.

It follows from Definition 2.3 that, if $x^*(\cdot)$ is an ET, then, for an initial position $x_0, t_0$, the approach problem of pencils $X[i, x_0, t_0, u\Delta]$ with terminal sets $M(i)$ is solvable in the presence of the undetermined parameter $i \in I$. In Sect.1 the set of such initial positions was called $K(I)$ and was constructed by solving $2^n - 1$ auxiliary problems on $(N-M)$ approach without undertermined parameters and in ordinary positional strategies without a memory. Starting from the fact that $x^*(\cdot)$ is an ET, we can also show that, for all intermediate positions $(x^*(t), t)$, as initial positions, the corresponding approach problem is solvable in the presence of an undetermined parameter $i \in I$. Thus, the entire ET $x^*(\cdot)$ lies in the set $K(I) \subset X \times T$.

For, let $x^*(\cdot)$ be an admissible trajectory (i.e., a trajectory which can approach as close as desired to a trajectory generated by a piecewise constant program $u'(t)$). Let $(x^*(t), t) \in K(I)$ for all $t \in [t_0, \vartheta]$. Let the set of strategies $u\Delta$ solve the corresponding approach problem with sufficiently small $\varepsilon$ for initial position $(x^*(t_0), t_0)$. We showed at the end of Sect.1 that this set of positional strategies with a memory about the trajectory may be constructed on the basis of ordinary positional strategies $v_\varepsilon^\delta \Delta_\varepsilon$, $\sigma \subseteq I$, which might be arbitrarily chosen inside the $\varepsilon'$-neighbourhood of $K(I)$, where $\varepsilon' > 0$ is sufficiently small. We now complete the definition of these positional strategies inside the $\varepsilon'$-neighbourhood of the set $K(I)$ by putting them equal to the program strategies, which approximate up to $\varepsilon'' < \varepsilon'$ the trajectory $x^*(\cdot)$ considered. It can be verified directly that the strategy $u\Delta = (u_1\Delta_1, \ldots, u_n\Delta_n)$ thus redefined satisfies all the conditions of Definition 2.3 and ensures satisfaction of the appropriate inequalities and inclusions up to $\varepsilon$.

We thus have:

*Theorem 2.1.* The admissible trajectory $x^*(\cdot)$ is an equilibrium trajectory if and only if it lies entirely in the set $K(I) \subset X \times T$.

This theorem is similar in form to the corresponding theorem of /3/, concerned with a non-antagonistic two-person positional game.

*Solution of a differential game of simple type.* We will consider a special case of differential game (2.1) in which the function $f$ has the form $f(u_1', \ldots, u_n')$, i.e., does not depend on $x$ and $t$, while the terminal functions $g_i$ are convex downwards. For this game we can construct the set $K(I)$ explicitly.

The solution is based on three auxiliary lemmas, concerning the approach problem of simple type

$$dx/dt = f(u', v'),\ x(t_0) = x_0,\ u' \in P,\ v' \in Q$$

The first player's task is to choose the positional strategy which ensures, whatever the second player's moves, the encounter at instant $\vartheta$ of all the trajectories with the convex terminal set $M$.

Let $W$ denote the maximum $u_*$-stable bridge in this approach problem.

*Lemma 2.1.* $W$ is a convex set.

*Lemma 2.2.* If $(x_0, t_0) \in W$, then the cone with vertex $(x_0, t_0)$ and base $M$ is a $u_*$-stable set and lies wholly in $W$.

*Lemma 2.3.* Set $W$ is given by the condition

$$W = \{x_0, t_0, | \forall \|q\| = 1 \quad \langle q, x\rangle + (\vartheta - t_0)\xi(q) \leqslant \sup_{x \in M} \langle q, x\rangle\}$$

$$\xi(q) = \min_{u' \in P} \max_{v' \in Q} \langle q, f(u', v')\rangle$$

We omit the proofs.

We return to the differential game of simple type. Following the general scheme outlined at the end of Sect.1, to construct the set $K(I)$ we have to construct the sets $K(\sigma)$ consecutively for all $\sigma \subseteq I$. If the sets $K(\sigma \setminus i)$ have been constructed, then, to construct $K(\sigma)$, we have to solve an $(N-M)$ approach problem, in which

$$N = \bigcap_{i \in \sigma} K(\sigma \setminus i) = N(\sigma), \quad M = \bigcap_{i \in \sigma} M(i) = M(\sigma)$$

Let $A(\sigma)$ be the set of initial positions for which this approach problem is solvable with terminal set $M(\sigma)$, but without phase constraints.

*Theorem 2.2.* $K(\sigma) = A(\sigma) \cap N(\sigma)$ and is a convex set $(K(\varnothing) = X \times T)$.

*Proof.* Assume that the theorem holds for all $\sigma' \subset \sigma$. We will show that it then also holds for $\sigma' = \sigma$. The fact that $K(\sigma) \subseteq A(\sigma) \cap N(\sigma)$ follows at once by the definition of $K(\sigma)$. Now let $(x_0, t_0) \in A(\sigma) \cap N(\sigma)$. By hypothesis, the sets $K(\sigma \setminus i)$ are convex. The set $A(\sigma)$ is convex by Lemma 2.1. Consequently, the cone with vertex $(x_0, t_0)$ and base $M(\sigma)$, first, lies entirely in the set $A(\sigma) \cap N(\sigma)$, and second, by Lemma 2.2, is a $u_*$-stable set. Thus, for the initial position $x_0, t_0$ the approach problem with terminal set $M(\sigma)$ inside set $N(\sigma)$ is solvable. The theorem is proved.

*Corollary 2.1.* $K(I) = \bigcap\limits_{\sigma \subseteq I} A(\sigma)$, where

$$A(\sigma) = \{x_0, t_0 \mid \forall \|q\| = 1\ \langle q, x_0\rangle + (\vartheta - t_0)\,\xi(\sigma, q) \leqslant \sup_{x \in M(\sigma)} \langle q, x\rangle\} \tag{2.6}$$
$$\xi(\sigma, q) = \min_{u_1', \ldots, u_n'} \max\{\langle q, f\rangle \mid f \in F(\sigma, u_1', \ldots, u_n')\}$$

The proof follows at once from Theorem 2.2. and Lemma 2.3.

To sum up, for games of simple type, Theorem 2.1 on the necessary and sufficient conditions for a trajectory to be an equilibrium trajectory can be restated as follows.

*Theorem 2.3.* The admissible trajectory $x^*(\cdot)$ is an equilibrium trajectory for the simple game if and only if it lies wholly in each of the sets $A(\sigma)$, defined by (2.6) and constructed for the terminal sets

$$M(\sigma) = \{x \mid g_i(x) \leqslant g_i(x^*(\vartheta)), \quad \forall i \in \sigma\}$$

As an example, consider the game

$$dx/dt = u_1' + u_2' + u_3', \quad x(t_0) = x_0, \quad t \in [t_0, \vartheta]$$
$$\|u_i'\| \leqslant r_i, \quad 0 \leqslant r_1 \leqslant r_2 \leqslant r_3$$
$$J_i = \langle g_i, x(\vartheta)\rangle \to \max$$

It is played on the plane. To be specific we assume that the players have no vector of common interests, i.e., there is no vector $g$ such that $\langle g, g_i\rangle > 0, \forall i$.

It can be shown that the function $\xi(\sigma, q)$ can be written for this problem in the form $\xi(\sigma)\|q\|$, where

$$\xi(\sigma) = \min_{i \in \sigma} \xi(i), \ \xi(1) = r_1 - r_2 - r_3$$
$$\xi(2) = r_2 - r_1 - r_3, \ \xi(3) = r_3 - r_1 - r_2$$

The set $A(\sigma)$ can be specified as

$$A(\sigma) = \{x_0, t_0 \mid \rho(x_0/(\theta - t_0), M(\sigma)) \leqslant -\xi(\sigma)\}$$
$$M(\sigma) = \{x \mid \langle g_i, x\rangle \leqslant \langle g_i, x^*(\vartheta)\rangle, \quad \forall i \in \sigma\}$$

where $\rho(\cdot, \cdot)$ is the distance between a point and a set, and $x^*(\cdot)$ is the trajectory studied at equilibrium.

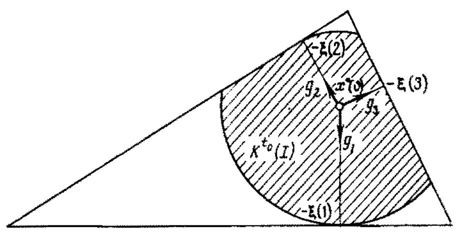


Fig.1

Thus the set $K(I)$ is here a cone with vertex at $(x^*(\vartheta), \vartheta)$, whose section by the instant $t_0 = \vartheta - 1$ is shown in Fig.1.

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Translated by D.E.B.